NOTES AND CORRESPONDENCE

On the Pressure Field in the Slope Wind Layer

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ABSTRACT

It has been suggested by some authors that the momentum equation for thermally driven slope flow should contain a horizontal pressure gradient term, in addition to the buoyancy term. It is shown that this suggestion is incorrect and leads to a spurious increase in along-slope forcing unless the vertical component of the perturbation pressure gradient is included as well. Along-slope accelerations due to the horizontal and vertical perturbation pressure gradients cancel each other exactly if the temperature perturbation is constant along the slope. Based on the concept of hydrostatic equilibrium perpendicular to the slope, the error associated with neglecting the vertical component of the pressure gradient, and the error due to the assumption of vertical hydrostatic equilibrium are evaluated. A revised conceptual diagram of the relationship between buoyancy and pressure forces within the slope wind layer is presented.

1. Introduction

In theoretical models of slope flows the primary forcing term in the momentum equation is the along-slope component of the buoyancy force (Prandtl 1942), with an additional along-slope pressure gradient term arising only if the temperature perturbation varies along the slope (Manins and Sawford 1979). This along-slope pressure gradient term due to thermal inhomogeneities has been called the ‘‘thermal wind term’’ by Mahrt (1982). Some authors, however, maintain that the horizontal pressure gradient needs to be included in the momentum equation in addition to the buoyancy term, even in the absence of along-slope perturbation temperature variations (Petkovsek 1982; Kossmann and Fiedler 2000). The notion that the horizontal hydrostatic pressure gradient due to the warmer (colder) air at the slope must cause some additional upslope (downslope) acceleration is intuitively appealing, and has entered some conceptual diagrams (e.g., Atkinson 1981, p. 253). The purpose of this note is to clarify the relationship between perturbation temperature, buoyancy, and perturbation pressure in the slope wind layer. Buoyancy and pressure gradient forces are derived in two different ways. The standard formulation, which uses a rotated coordinate system (s, n), is compared with the ‘‘mixed’’ approach, which refers to both the (s, n) and the (x, z) systems, and which has been used by some authors (Atkinson 1981; Petkovsek 1982; Kossmann and Fiedler 2000). It is shown that a spurious up-slope acceleration is created by taking into account only the horizontal component of the perturbation pressure gradient, rather than the full perturbation pressure gradient. Moreover, due to the approximate hydrostatic balance that exists perpendicular to slope, the along-slope pressure gradient term vanishes identically when there are no thermal inhomogeneities along the slope (Mahrt 1982).

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2. Pressure perturbation and buoyancy

In a standard Cartesian coordinate system, horizontal and vertical accelerations due to the pressure field and gravity are given by

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1}$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{2}$$

where the usual notation is used. We define a horizontally homogeneous basic state at rest and in hydrostatic equilibrium. Density variations are neglected except in the nominator of the buoyancy term, and expressed in terms of potential temperature perturbations corresponding to the Boussinesq approximation for shallow flow. Thus, we obtain

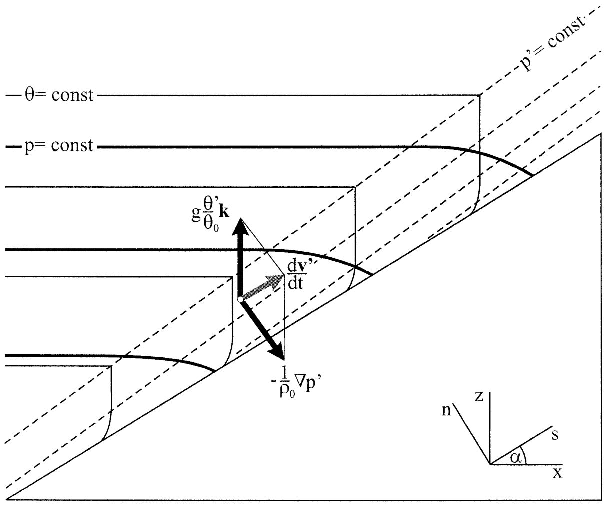

FIG. 1. Conceptual diagram of the combined effect of buoyancy and perturbation pressure in thermally driven slope flow. Shown are isolines of potential temperature (thin lines), perturbation pressure (dashed lines), and pressure (bold lines) for the case of a daytime up-slope flow with a thermal perturbation that increases along the slope. Due to this increase, the perturbation pressure gradient gives a small positive contribution to along-slope acceleration in this case. Perpendicular to the slope, the flow is quasi-hydrostatically balanced. In the special case of along-slope thermal homogeneity the dashed lines would be parallel to the slope, and the perturbation pressure gradient would be exactly perpendicular to the terrain.

$$\frac{du'}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}, \tag{3}$$

$$\frac{dw'}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} + g\frac{\theta'}{\theta_0}, \tag{4}$$

where the prime denotes perturbation quantities, and the basic state is indicated by the subscript 0.

a. Buoyancy and pressure terms in rotated coordinates

The essence of the problem can be discussed by considering the case of a slope with constant inclination angle α (Fig. 1). Rotated coordinates are denoted by (s, n), where s is up-slope distance, and n is the distance perpendicular to the slope. The corresponding velocity components are u_s and w_n. Since the rotated coordinate system is still orthogonal, no metric terms appear, and the equations equivalent to (3) and (4) take the form

$$\frac{du'_s}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial s} + g\frac{\theta'}{\theta_0}\sin\alpha, \tag{5}$$

$$\frac{dw'_n}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial n} + g\frac{\theta'}{\theta_0}\cos\alpha \tag{6}$$

(e.g., Egger 1981). The only difference compared to the (x, z) system is that gravity now has components in both coordinate directions. In vector notation both (3), (4) and (5), (6) can be written

$$\frac{d\mathbf{v}'}{dt} = -\frac{1}{\rho_0}\nabla p' + g\frac{\theta'}{\theta_0}\mathbf{k}. \tag{7}$$

b. "Mixed" formulation

Alternatively, the along-slope acceleration can be expressed as the sum of the horizontal acceleration (3) multiplied by the cosine of the slope angle, and the vertical acceleration (4) multiplied by the sine of the slope angle:

$$\frac{du'_s}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}\cos\alpha - \left(\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - g\frac{\theta'}{\theta_0}\right)\sin\alpha. \tag{8}$$

We refer to this form as mixed because an acceleration in the (s, n) system is expressed in terms of derivatives in the (x, z) system. Because of

$$\frac{\partial p'}{\partial s} = \frac{\partial p'}{\partial x}\cos\alpha + \frac{\partial p'}{\partial z}\sin\alpha, \tag{9}$$

the resulting equation is identical to (5), with the two pressure gradient terms on the rhs of (8) combining to give the first term on the rhs of (5).

Petkovsek (1982) and Kossmann and Fiedler (2000) employ the mixed approach, but instead of the full vertical acceleration [the term in parentheses in (8)], they take into account only the buoyancy term. Their equations for along-slope acceleration are equivalent to

$$\frac{du'_s}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}\cos\alpha + g\frac{\theta'}{\theta_0}\sin\alpha. \tag{10}$$

The vertical component of the perturbation pressure gradient is missing, which leads to an incorrect along-slope acceleration. As shown below, the error can be large and will usually lead to an overestimation of the along-slope acceleration.

Fleagle (1950), in his theoretical study of slope winds, also refers to the horizontal pressure gradient as a forcing term. However, he does not include buoyancy *and* the horizontal pressure gradient in his momentum equation but expresses the latter in terms of the temperature perturbation by assuming vertical hydrostatic equilibrium. The effect of this assumption on along-slope acceleration is discussed below. It should be mentioned that Atkinson (1981), contrary to what is suggested by his conceptual diagram, does not consider buoyancy and the horizontal pressure gradient as two independent terms but rather follows the derivation of Fleagle (1950).

3. Quasi-hydrostatic equilibrium

Up to this point no assumptions in addition to those of the shallow Boussinesq approximation and constant slope angle have been made. In order to get a more specific estimate of the error involved in using (10), we take into account the fact that slope flows are close to a state of hydrostatic equilibrium in the direction perpendicular to the ground. Mahrt (1982) introduced the term ‘‘quasi-hydrostatic’’ to distinguish this particular balance from the more familiar vertical hydrostatic equilibrium. Defining an along-slope length scale L, and a flow depth scale H, it can be shown that the ratio of the acceleration terms to the buoyancy term in the w'_n-equation of motion (6) is of the order of

$$O\left(\frac{dw'_n}{dt}\bigg/ g\frac{\theta'}{\theta_0}\cos\alpha\right) = \frac{H}{L}\tan\alpha \tag{11}$$

(Mahrt 1982). This ratio is $\ll$1 if the slope wind layer has the characteristics of a boundary layer that is ‘‘thin’’ compared to the radius of curvature of the topographic profile. This is the case for many real slopes. According to (11), dynamic pressure effects become significant only in very steep terrain, or near abrupt changes in terrain inclination, such as sharp ridges. It should be noted that we do not consider here the initial, transient response of the atmosphere to localized diabatic warming (cooling), namely the formation of a positive (negative) pressure perturbation in the affected region, and the generation of thermal compression waves (Nicholls and Pielke 1994). As is customary in studies of thermally driven slope flows we assume that the fast, compressible mass adjustment has already taken place, and that pressure perturbations are small outside the slope wind layer.

If (11) is small, the momentum equation (6) perpendicular to the slope reduces to quasi-hydrostatic equilibrium

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial n} = g\frac{\theta'}{\theta_0}\cos\alpha. \tag{12}$$

Important consequences of (12) are (a) the pressure perturbation field can be derived directly from the temperature perturbation field (and vice versa), regardless of the flow; (b) if the temperature perturbation is uniform in along-slope direction, the along-slope perturbation pressure gradient vanishes, and only the buoyancy term appears on the rhs of (5). As noted by Mahrt (1982), virtually all theoretical slope wind models are explicitly or implicitly based on quasi-hydrostatic equilibrium.

Figure 1 illustrates Eq. (7) under the quasi-hydrostatic equilibrium assumption for a daytime slope wind layer that increases in depth along the slope. The primary along-slope forcing is due to buoyancy. The perturbation pressure gradient gives a small up-slope contribution in this case because the temperature perturbation has been assumed to increase going up the slope. If the positive temperature perturbation would decrease along the slope, the pressure gradient would give a negative contribution. In the direction perpendicular to the slope, buoyancy and the pressure gradient cancel each other [cf. (12)] so the resulting acceleration vector is parallel to the surface.

There are two special cases of interest. First, if the thermal anomaly is constant along the slope, the pressure gradient term is exactly perpendicular to the ground and does not contribute to along-slope acceleration. This applies to slope flows in local equilibrium, which have been studied analytically by Prandtl (1942), and numerically by Schumann (1990). Analysis of observational data obtained during the Vertical Transport and Mixing (VTMX) program (Doran et al. 2002) indicates that katabatic flow close to local equilibrium does indeed occur, even on rather low-angle slopes (Haiden and Whiteman 2002). Second, the opposing perturbation pressure gradient can become so large that it essentially reduces the net along-slope forcing to zero. One example would be a katabatic flow layer that increases in depth down the slope at such a rate that its top becomes more or less horizontal, as happens when downslope flows converge into a valley or basin to form cold air pools.

Using (12), and relationships analogous to (9) between the differentials in both coordinate systems, we can express incorrect Eq. (10) in the form

$$\frac{du'_s}{dt} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial s}\cos^2\alpha + g\frac{\theta'}{\theta_0}\sin\alpha(1 + \cos^2\alpha). \tag{13}$$

Comparison with (8) shows that for small slope angles ($\alpha < 10°$), where $\cos^2\alpha \approx 1$, this is roughly equivalent to double-counting the along-slope component of buoyancy. For steeper slopes, the error depends on the relative importance of the two forcing terms.

The slope wind layer cannot be at the same time in vertical hydrostatic equilibrium *and* quasi-hydrostatic equilibrium. What happens if a model that is based on hydrostasy in the vertical,

$$\frac{1}{\rho_0}\frac{\partial p'}{\partial z} = g\frac{\theta'}{\theta_0}, \tag{14}$$

rather than on (12), is used to compute along-slope acceleration? Fleagle (1950), for example, assumed vertical hydrostatic equilibrium in his theoretical study of katabatic flows. If hydrostasy is assumed in numerical simulations of thermally driven flows in complex terrain it is also usually meant to exist in the vertical. For a slope flow that is to first-order parallel to the surface, that is, in which $w'_n \ll u'_s$, the associated error can be quantified as follows. We compute the horizontal acceleration from the horizontal momentum equation (3), and relate the vertical velocity diagnostically to u' via the kinematic boundary condition

$$w' = u'\tan\alpha. \tag{15}$$

Using (14), (15), and (9), the hydrostatic along-slope acceleration can be expressed

$$\left(\frac{du'_s}{dt}\right)_{HS} \equiv \frac{du'}{dt}\cos\alpha + \frac{dw'}{dt}\sin\alpha$$
$$= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}(\cos\alpha + \tan\alpha\,\sin\alpha)$$
$$= \left(-\frac{1}{\rho_0}\frac{\partial p'}{\partial s} + g\frac{\theta'}{\theta_0}\right)(1 + \tan^2\alpha). \quad (16)$$

Comparison with (5) shows that the assumption of vertical hydrostatic balance creates a spurious along-slope forcing proportional to $\tan^2\alpha$. The error is negligible for $\alpha < 10°$ but increases rapidly for steeper slopes. Since $\tan\alpha$ is the ratio of the vertical to the horizontal cross section of the slope wind layer, this is a special case of the general rule that effects associated with deviations from vertical hydrostatic equilibrium increase with the square of the aspect ratio of the flow (e.g., Pielke 1984).

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